1. Tools

Def.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a mapping.
We now define:

1. $\phi : J \to \mathbb{R}$ is convex, iff
   $\forall x, y \in J \quad \forall t \in [0;1] \quad \phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$

2. Let $\phi(J) \subseteq \mathbb{R}_+$.
   $\phi : J \to \mathbb{R}$ is logarithmically convex, iff
   $\ln(\phi) : J \to \mathbb{R}$ is convex

Rem.: Let $\phi(J) \subseteq \mathbb{R}_+$.
Because $\exp : \mathbb{R} \to \mathbb{R}$ is convex and monotonically increasing, we get the following:

$(\phi : J \to \mathbb{R}$ is logarithmically convex$) \Rightarrow$
$(\phi : J \to \mathbb{R}$ is convex$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$(\phi' : J \to \mathbb{R}$ is monotonically increasing$)$

Theo.:

Pre.: Let J be a non-empty interval of $\mathbb{R}$.
Let $\phi : J \to \mathbb{R}$ be a 2-times differentiable mapping.

Ass.: $(\phi : J \to \mathbb{R}$ is convex$) \Leftrightarrow$
$\phi'' \geq 0$

2. Gamma-Function

The Gamma-Funktion $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$ is for all $\alpha \in \mathbb{R}_+$ defined through the absolutely convergent integral

$$\Gamma(\alpha) := \underbrace{\int_0^\infty \tau^{\alpha-1} \cdot e^{-\tau} \, d\tau}_{>0}$$

From literature we have:

$$\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is analytically} \tag{1}$$

$$\forall \alpha \in \mathbb{R}_+ \quad \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha) \tag{2}$$

$$\forall k \in \mathbb{N}_0 \quad \Gamma(k+1) = k! \tag{3}$$

$$\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is logarithmically convex (and ergo convex)} \tag{4}$$

$$\Gamma(1) = 1 \text{ and } \Gamma(2) = 1 \tag{5}$$

With (4) and (5) we have:

$$\Gamma \mid [2; \infty[\text{ is monotonically increasing} \tag{6}$$

We now define a mapping $\gamma :]-1; \infty[\rightarrow \mathbb{R}$ through

$$\forall u \in]-1; \infty[\quad \gamma(u) := \Gamma(u+1)$$

Then we have with (2):

$$\forall v \in]-1; \infty[\quad \gamma(v+1) = (v+1)\,\gamma(v) \tag{7}$$

In addition we have with (6):

$$\gamma \mid [1; \infty[\text{ is monotonically increasing} \tag{8}$$

3. A Look at Taylor-Series

Let $\left(a_n\right)_{n\in\mathbb{N}_0}$ be a sequence in $\mathbb{R}$.
Let $\rho \in \mathbb{R}_+ \cup \{\infty\}$ be the radius of convergence of $f := \sum_{n=0}^{\infty} a_n x^n$.
Let $J :=]0; \rho[$.
Now we define for $\tilde{\alpha} \in]-1; \infty[$ the function $p_{\tilde{\alpha}} : J \to \mathbb{R}$ through

$$\forall t \in J \quad p_{\tilde{\alpha}}(t) := \sum_{n=0}^{\infty} a_n \frac{\gamma(n + \tilde{\alpha})}{\gamma(n + \tilde{\alpha} + 1)} t^{n+\tilde{\alpha}+1}$$

Then we have (Cave (7) and (8)!) for all $\tilde{\alpha} \in]-1; \infty[$:

$p_{\tilde{\alpha}} : J \to \mathbb{R}$ is well-defined and differentiable

and

$$\begin{aligned}\forall t \in J \quad \left(p_{\tilde{\alpha}}\right)'(t) &= \sum_{n=0}^{\infty} a_n \frac{\gamma(n + \tilde{\alpha})}{\gamma(n + \tilde{\alpha} + 1)} \left(x^{n+\tilde{\alpha}+1}\right)'(t) = \\ &= \sum_{n=0}^{\infty} a_n t^{n+\tilde{\alpha}} = \\ &= \left(\sum_{n=0}^{\infty} a_n t^n\right) t^{\tilde{\alpha}} = \\ &= t^{\tilde{\alpha}} f(t)\end{aligned}$$

4. Literature

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